given $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ filtered probability space

Def (Itô process) Given a m-dimensional BM $\{B_t\}_{t\geq 0}$ and coefficients (measurable)

$$b : [0,+\infty) \times \Omega \longrightarrow \mathbb{R}^n \qquad \sigma : [0,+\infty)\times\Omega \longrightarrow \mathbb{R}^{n\times m}$$

given a starting point $x_0 \in \mathbb{R}^n$ we define an Itô process as a process $\{X_t\}_{t\geq 0}$ such that

$$X_t = X_0 + \int_0^t b(s,\omega)\,ds + \int_0^t \sigma(s,\omega)\,dB_s \quad (1)$$

We will use the differential notation

$$dX_t = b\,dt + \sigma\,dB_t \quad ; X_0 = x_0 \quad (2)$$

Def $\boxed{\text{Itô diffusion}}$ ~~It is a stochastic process~~

~~$\{X_t\}_{t\geq 0} = \{X_t^x\}_{t\geq 0}$ where $x \in \mathbb{R}^n$ is the starting point~~

~~such that~~

~~$X^t = x + \int_0^t$~~

Given coefficients

$$b : \Omega \longrightarrow \mathbb{R}^n \quad \sigma : \Omega \longrightarrow \mathbb{R}^{n\times m}$$

and a starting point $x \in \mathbb{R}^n$, an Itô diffusion is a process $\{X_t\}_{t\geq 0} \equiv \{X_t^x\}_{t\geq 0}$ such that

$$X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s \quad (3)$$

or

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \qquad X_0 = x \quad (4)$$

$\boxed{\text{Theo}}$ (existence of solutions of stochastic diff. eq.)

A stochastic diff equation (SDE) of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \ ; \ X_0 = Z$$

has a unique solution that is t-continuous up to a given time T as long as $|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|) \ \forall x, \forall t \in [0,T]$

and $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y| \ \forall x, y, \forall t \in [0,T]$

and Z is a random variable independent from $\mathcal{F}_\infty$, where $\mathcal{F}_\infty = \sigma(\{B_s \mid s \geq 0\})$ and $E[|Z|^2] < +\infty$

$\boxed{\text{Simple example}}$ A Brownian motion is a Itô Process with $b \equiv 0$, $\sigma \equiv 1$ $\quad B_t^x = x + \int_0^t 1 \, dB_s$

$\boxed{\text{Def}}$ (Stopping time) A random variable $\tau : \Omega \to [0, +\infty]$ is a stopping time w.r.t. a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if $\{\tau \leq t\} \in \mathcal{F}_t \ \forall t \geq 0$. Let $\mathcal{F}_\infty$ be the smallest σ-algebra containing $\mathcal{F}_t$ for all $t \geq 0$.

Then we define by $\mathcal{F}_\tau$ the σ-algebra of sets $A \in \mathcal{F}_\infty$ such that

$A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$

~~Let $X_\tau: \Omega \to \mathbb{R}$ a random variable measurable w.r.t. $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$. Then we define~~

Let $\{X_t\}_{t\geq 0}$ be a process. (i.e. $X_t : \Omega \to \mathbb{R}^n$ measurable w.r.t. $\mathcal{F}$ and $\mathcal{B}(\mathbb{R}^n)$). Then we define

$$X_\tau : \Omega \to \mathbb{R} \; ; \; X_\tau(\omega) = \begin{cases} X_{\tau(\omega)}(\omega) & \text{if } \tau(\omega) < +\infty \\ 0 & \text{if } \tau(\omega) = +\infty \end{cases}$$

X_τ is a random variable measurable w.r.t. $\mathcal{F}_\tau$ and $\mathcal{B}(\mathbb{R}^n)$

Important example Let $A \subseteq \mathbb{R}^n$ be a Borel set.
Let X_t be an Itô process (Itô diffusion too). Then $\tau(\omega) = \inf\{t \geq 0 \mid X_t(\omega) \notin A\}$ is a stopping time

Strong Markov property Let $\{X_t^x\}_{t\geq 0}$ be a Itô diffusion
Let τ be a stopping time. Then

$$E\left[f(X^x_{\tau+h}) \,\middle|\, \mathcal{F}_\tau\right] = E\left[f\left(X_h^{X_\tau^x}\right)\right] \quad \forall h > 0$$

not $E[f(X^x_t)] =: E^x[f(X_t)]$

Def (Infinitesimal generator) Let $X = \{X_t\}_{t\geq 0}$ be a Itô diffusion in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function; $f \in C^2(\mathbb{R}^n)$.
The infinitesimal generator of X is the operator $\mathcal{A}$ defined by

$$\mathcal{A}f(x) = \lim_{t \to 0} \frac{E^x[f(X_t)] - f(x)}{t}$$

Example: if $\{X_t\}_{t\geq 0} = \{B_t\}_{t\geq 0}$ then $\mathcal{A}f(x) = \frac{1}{2}\Delta f(x)$.

$\boxed{\text{Itô's formula}}$ Let $dX_t = b_t\,dt + \sigma\,dB_t$ be a Itô process.

Let $f = f(t,x) \in C^2(\mathbb{R}\times\mathbb{R}^n)$. Then

$$df(t,X_t) = \Big[(\partial_t f)(t,X_t) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{i,j}(t,X_t)\cdot(\partial^2_{x_i x_j} f)(t,X_t)\Big]dt + \sum_{i=1}^{n}(\partial_{x_i} f)(t,X_t)\,dX^i_t$$

$$X_t = (X^1_t, X^2_t, \dots, X^n_t)$$

$\boxed{\text{Dynkin's formula}}$ Let $dX_t = b\,dt + \sigma\,dB_t$ be a Itô process

Let $f \in C_0^2(\mathbb{R}^n)$. Let τ be a stopping time w.r.t. $\{\mathcal{F}_t\}$ and such that $E[\tau] < +\infty$. Assume that $X(t,\omega) \in \text{Supp}(f)$

then
$$E^x[f(X_\tau)] = f(x) + E^x\Big[\int_0^\tau Af(X_s)\,ds\Big]$$

Where
$$Af(X_s) = \sum_{i=1}^{n} b_i(s,X_s)\frac{\partial f}{\partial x_i}(X_s) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{i,j}(s,X_s)\frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)$$

A is the infinitesimal generator of the process $\{X_t\}$

$\boxed{\text{Proof}}$ after proving the characterization of the infinitesimal generator it is easy to prove Dynkin's formula starting from Itô's formula, because by Itô's formula we have that

$$f(X_\tau) = f(x) + \int_0^\tau \frac{1}{2} \sum_{i,j=1}^{m} (\sigma\sigma^T)(s, X_s)\, \partial^2_{x_i x_j} f(s, X_s)\, ds \; +$$

$$+ \int_0^\tau \sum_{i=1}^{m} b(s, X_s)\, \partial_{x_i} f(s, X_s)\, ds \; +$$

$$+ \int_0^\tau \sum_{i=1}^{m} \partial_{x_i} f(s, X_s) \cdot \left(\sigma \cdot (s, X_s)\, dB_s\right)^i$$

$$\Rightarrow E^x(X_\tau) = f(x) + E^x\left[\int_0^\tau A f(X_s)\, ds\right] \quad + 0$$